Lecture 29

15.8 & 15.9 - Cylindrical and Spherical Coordinates

Cylindrical coordinates are polar coordinates with z added in. The coordinates are (r, θ, z) which relate to Cartesian coordinates by:

$$\boxed{x = r\cos\theta,\ y = r\sin\theta,\ z = z},$$

and in the reverse direction:

$$\boxed{r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x},\ z = z}$$

Ex: a) Write the point with cylindrical coordinates $(4, \frac{\pi}{3}, -2)$ in rectangular coordinates.

b) Write the point with rectangular coordinates $(-2, 2\sqrt{3}, 3)$ in cylindrical coordinates.

Sol: a) $(r, \theta, z) = (4, \frac{\pi}{3}, -2)$

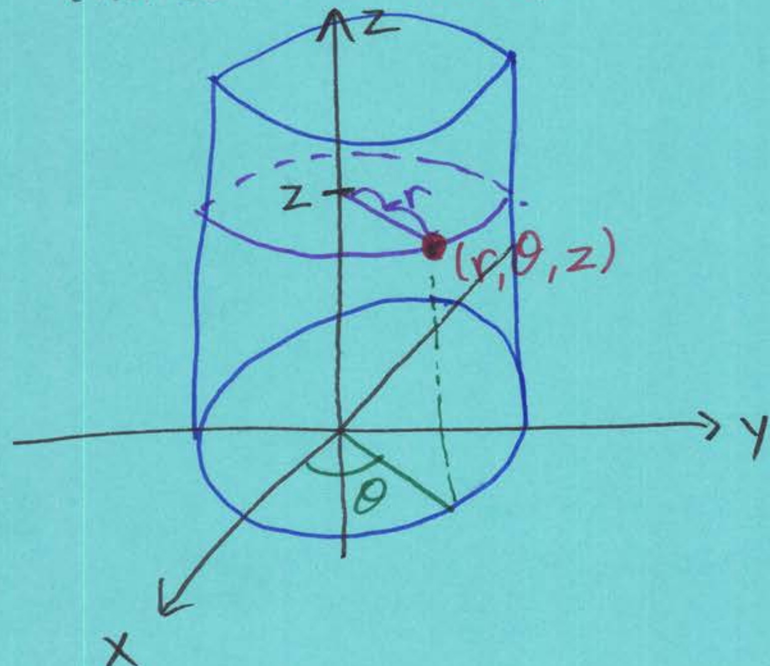
$$(x,y,z) = (r\cos\theta, r\sin\theta, z) = (4\cos\tfrac{\pi}{3}, 4\sin\tfrac{\pi}{3}, -2) = (2, 2\sqrt{3}, -2)$$

b) $(x,y,z) = (-2, 2\sqrt{3}, 3)$. $r = \sqrt{x^2+y^2} = \sqrt{4+12} = 4$

$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3} \Rightarrow \theta = \frac{2\pi}{3} + n\pi$, take $\theta = \frac{2\pi}{3}$.

So, $(r, \theta, z) = (4, \frac{2\pi}{3}, 3)$.

Visually, the point with cylindral coordinates (r, θ, z) is the point a distance r from the z axis, makes an angle of θ with the positive x-axis, and is a distance z above the xy-plane (negative z means below). You can also think of it as being on the cylinder of radius r (hence the name):



The point of introducing these is to do integration. So, what is dV in cylindrical coordinates?

$dV = dx\,dy\,dz$ (or some other iteration), and

$dx\,dy = r\,dr\,d\theta$, so $dV = r\,dr\,d\theta\,dz$.

Revisit an example from Wednesday:

Ex: Find the volume of the solid bounded by $z = x^2 + y^2$ and $z = 4$.

Sol: In cylindrical coordinates $z = x^2 + y^2 = r^2$.

If we integrate z first, we look at the shadow in the xy-plane, which we determined was the disk of radius 2: $x^2 + y^2 \leq 4$. In polar coordinates, this is $r^2 \leq 4 \iff r \leq 2$.

So, $\mathrm{Vol}(E) = \iiint_E 1\, dV$

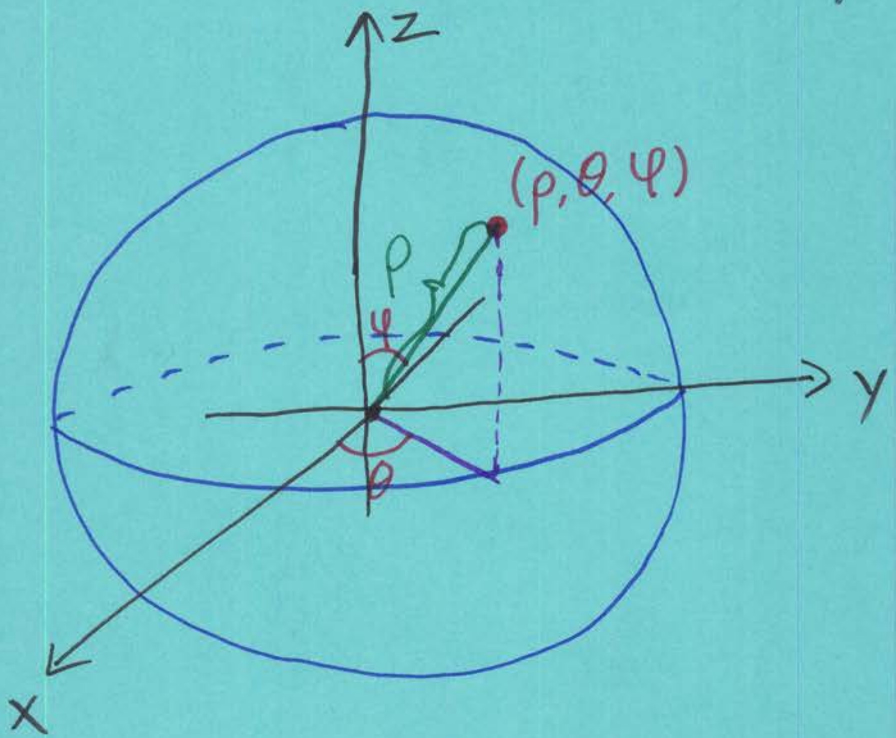
$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta = \int_0^{2\pi} \left(2r^2 - \tfrac{1}{4} r^4\right)\Big|_0^2 d\theta$$

$$= \int_0^{2\pi} (8 - 4) \, d\theta = 8\pi$$

$\square$

As cylinders can be used to give coordinates on $\mathbb{R}^3$, spheres can do the same. Visually:



The point with spherical coordinates (ρ, θ, φ) is on the sphere of radius ρ, makes an angle of θ with the positive x-axis, and an angle of φ with the positive z-axis. We take $\rho \geq 0$ and we only need $0 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$. The conversion formulas are

Spherical $\rightarrow$ Rectangular

$$\boxed{x = \rho \cos\theta \sin\varphi, \; y = \rho \sin\theta \sin\varphi, \; z = \rho \cos\varphi}$$

Rectangular $\rightarrow$ Spherical $\boxed{\rho^2 = x^2 + y^2 + z^2}$

Use the equations for x, y, & z above to find θ & φ.

Ex: a) The point $(3, \frac{\pi}{2}, \frac{3\pi}{4})$ is given in spherical coordinates. Find the rectangular coordinates.

b) The point $(-1, 1, -\sqrt{2})$ is given in rectangular coordinates. Find the spherical coordinates.

Sol: a) $(\rho, \theta, \varphi) = (3, \frac{\pi}{2}, \frac{3\pi}{4})$

$x = \rho\cos\theta\sin\varphi = 3\cos\frac{\pi}{2}\sin\frac{3\pi}{4} = 0$

$y = \rho\sin\theta\sin\varphi = 3\sin\frac{\pi}{2}\sin\frac{3\pi}{4} = \frac{3\sqrt{2}}{2}$

$z = 3\cos\frac{3\pi}{4} = \frac{-3\sqrt{2}}{2}$

Rectangular coordinates: $(x, y, z) = (0, \frac{3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2})$

b) $(x, y, z) = (-1, 1, -\sqrt{2})$

$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 1 + 2} = 2$ $\qquad 0 \le \varphi \le \pi$

$z = \rho\cos\varphi \Rightarrow \cos\varphi = \frac{z}{\rho} = \frac{-\sqrt{2}}{2} \Rightarrow \varphi = \frac{3\pi}{4}$ or $\cancel{\frac{5\pi}{4}}$

$y = \rho\sin\theta\sin\varphi \Rightarrow \sin\theta = \frac{y}{\rho\sin\varphi} = \frac{1}{2\sin\frac{3\pi}{4}} = \frac{1}{2(\frac{\sqrt{2}}{2})} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\Rightarrow \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

Check with eqn for x:

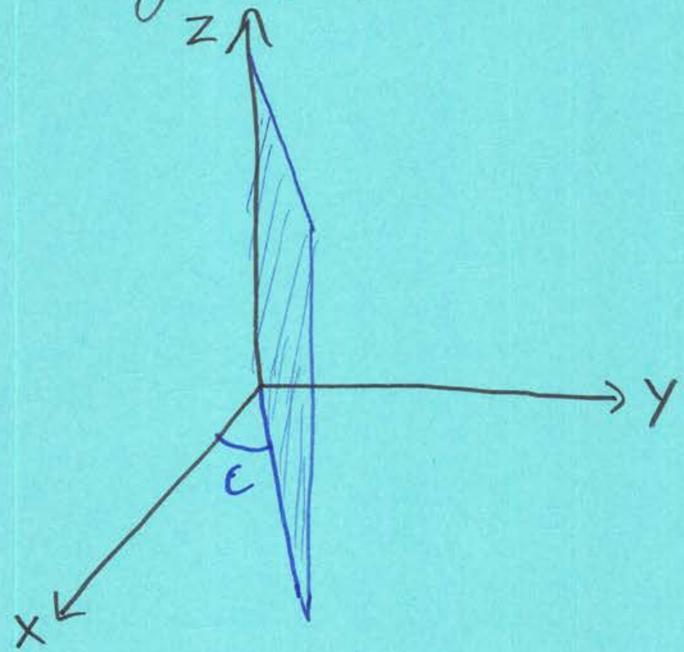
$-1 = 2\cos\theta\sin\frac{3\pi}{4} = \sqrt{2}\cos\theta \Rightarrow \cos\theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$

Spherical coordinates: $(\rho, \theta, \varphi) = (2, \frac{3\pi}{4}, \frac{3\pi}{4})$ ◇

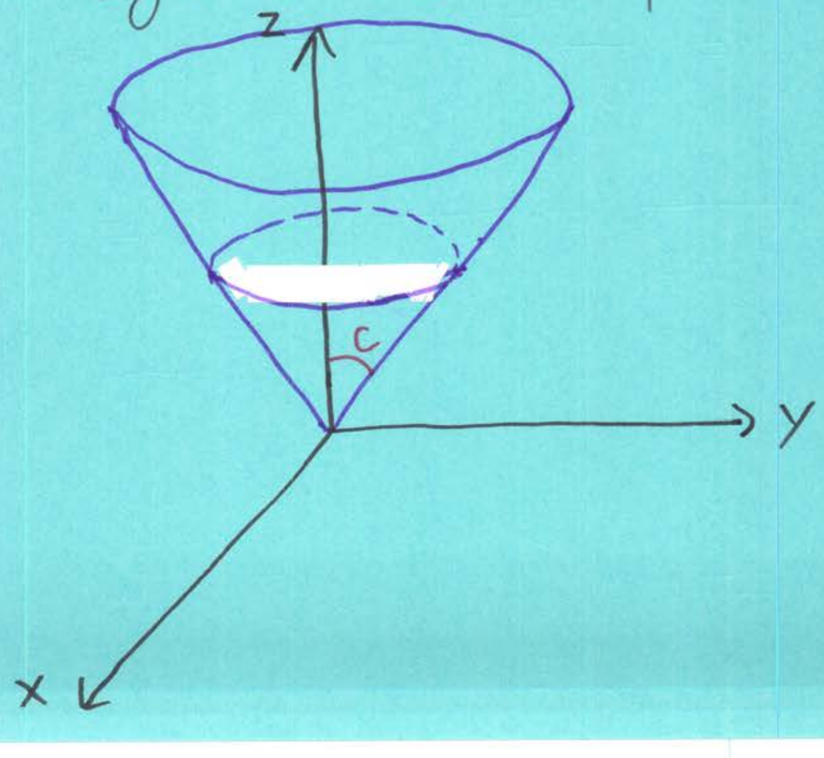
Some examples of surfaces in spherical coordinates:

i) $\rho = c$ is a sphere of radius c.

ii) $\theta = c$ is a half-plane starting at the z-axis, and making an angle c with the positive x-axis:



iii) $\varphi = c$ is a cone, opening along the z-axis and making an angle c with the positive z-axis:



Of course, I've introduced these coordinates to make some integrations easier. So, what is dV in spherical coordinates? It'll be quite a pain to derive this, so I'll leave that to the book: We have: $\boxed{dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi.}$ Let's see an example of this in action:

Ex: Find the volume of the region inside the sphere $x^2+y^2+z^2=4z$ and above the cone $z=\sqrt{\frac{1}{3}(x^2+y^2)}$.

Sol: The sphere in spherical coordinates is

$$\rho^2 = x^2+y^2+z^2 = 4z = 4\rho\cos\varphi$$

$$\Rightarrow \rho = 4\cos\varphi$$

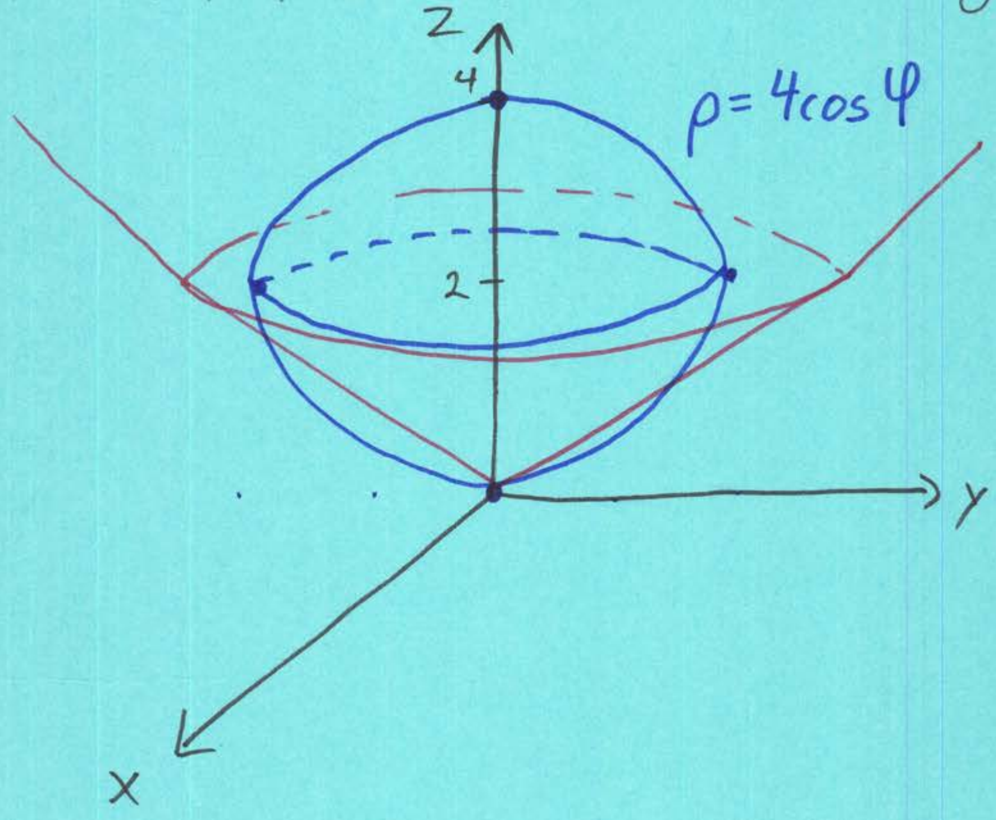
The cone:

$$\cancel{\rho}\cos\varphi = z = \sqrt{\tfrac{1}{3}(x^2+y^2)} = \sqrt{\tfrac{1}{3}(\rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi)}$$

$$= \sqrt{\tfrac{1}{3}\rho^2\sin^2\varphi} = \tfrac{1}{\sqrt{3}}\cancel{\rho}\sin\varphi$$

$$\Rightarrow \tan\varphi = \sqrt{3} \Rightarrow \varphi = \tfrac{\pi}{3}.$$

If we write the sphere in standard form, we have $x^2+y^2+(z-2)^2=4$, the sphere of radius 2 with center $(0,0,2)$. A sketch of the region is:



Here, we think in terms of where the radius starts and ends for the bounds on ρ. We can see then that $0 \le \rho \le 4\cos\varphi$. θ goes all the way around, so $0 \le \theta \le 2\pi$. φ starts at the top and ends on the cone, so $0 \le \varphi \le \frac{\pi}{3}$. Thus

$$\text{Vol} = \iiint_E 1\, dV = \int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^{4\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$-\tfrac{1}{16}-(-1)$$

$$= \int_0^{\frac{\pi}{3}}\int_0^{2\pi} \frac{4^3}{3}\cos^3\varphi \sin\varphi \, d\theta \, d\varphi = \frac{128\pi}{3}\int_0^{\frac{\pi}{3}} \cos^3\varphi \sin\varphi \, d\varphi = \frac{32\pi}{3}\left(-\cos^4\varphi\right)\Big|_0^{\frac{\pi}{3}}$$

$$= 10\pi \quad \diamond$$

One of my favorite spherical coordinates problems is:

Ex: Show $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz=2\pi$.

Sol: This is an integral over all of $\mathbb{R}^3$! In spherical coordinates, $\mathbb{R}^3=\{(\rho,\theta,\varphi)\,|\,0\le\rho<\infty,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi\}$

So, the integral becomes

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\rho e^{-\rho^2}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi=\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\rho^3e^{-\rho^2}\sin\varphi\,d\rho\,d\theta\,d\varphi$$

Let's worry about the ρ integral first:

$$\int_0^{\infty}\rho^3e^{-\rho^2}\,d\rho \underset{\substack{u=\rho^2\\ dv=\rho e^{-\rho^2}d\rho}}{\overset{\text{IBP}}{=\!=}} -\frac{1}{2}\rho^2e^{-\rho^2}\Big|_0^{\infty}+\int_0^{\infty}\rho e^{-\rho^2}\,d\rho$$

$$=\left(-\frac{1}{2}\rho^2e^{-\rho^2}-\frac{1}{2}e^{-\rho^2}\right)\Big|_0^{\infty}=\frac{-1}{2}\lim_{R\to\infty}\left[\left(R^2e^{-R^2}+e^{-R^2}\right)-(0+1)\right]$$

$=-\frac{1}{2}(-1)=\frac{1}{2}$. So the integral is

$$\int_0^{\pi}\int_0^{2\pi}\frac{1}{2}\sin\varphi\,d\theta\,d\varphi=\int_0^{\pi}\pi\sin\varphi\,d\varphi=-\pi\cos\varphi\Big|_0^{\pi}=-\pi(-1-(-1))=2\pi \quad\diamond$$